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Abstract

We investigate three-body motion in three dimensions under the interaction potential proportional to r^α ($\alpha \neq 0$) or $\log r$, where r represents the mutual distance between bodies, with the following conditions: (I) the moment of inertia is a non-zero constant, (II) the angular momentum is zero and (III) one body is on the centre of mass at an instant. We prove that the motion which satisfies conditions (I)–(III) with equal masses for $\alpha \neq -2, 2, 4$ is impossible. And motions which satisfy the same conditions for $\alpha = 2, 4$ are solved explicitly. Shapes of these orbits are not figure-eight and these motions have collision. Therefore, the moment of inertia for figure-eight choreography for $\alpha \neq -2$ is proved to be inconstant along the orbit. We also prove that the motion which satisfies conditions (I)–(III) with general masses under the Newtonian potential $\alpha = -1$ is impossible.

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1. Introduction

In 1970s, Saari formulated a conjecture [1, 2], which is now called 'Saari's conjecture': *In the n-body problem under the Newtonian gravity, if the moment of inertia is constant then the motion must be a relative equilibrium.* Recently, three-body choreography, equal mass three-body periodic motion on a planer closed curve on which each body chases each other, was found by Moore [3], Chenciner, Montgomery [4] and Simó [5, 6]. This motion is now called 'three-body figure-eight choreography'. Simó noted that the moment of inertia was not constant on figure-eight solution for the Newtonian potential, despite the relative variation

along the orbit being small [7]. Inconstancy of the moment of inertia of figure-eight solution is consistent with Saari's conjecture.

On the other hand, it is well known that in the n-body problem under the attractive potential proportional to r^{-2}, where r is the mutual distance between bodies, the moment of inertia I for any periodic motion must be constant. This is because the second derivative of the moment of inertia with respect to time under this potential yields the Lagrange–Jacobi identity $d^2 I/dt^2 = 2E$, where E represents the total energy. Integrating this equation, we get $I = Et^2 + c_1 t + c_2$ with integration constants c_1 and c_2. For any periodic motion under this potential, therefore, the total energy must be zero and the moment of inertia must be constant.

Numerical evidence of existence of three-body figure-eight choreography is known under the attractive interaction potential proportional to r^α ($\alpha \neq 0$) with $\alpha < 2$ or $\log r$ [3]. Then, Chenciner formulates a problem [8]: *show that the moment of inertia of figure-eight choreography stays constant only when $\alpha = -2$*. We call this problem Saari–Chenciner's problem. In this paper, we solved Saari–Chenciner's problem.

Actually, we investigated the three-body motion in three dimensions under the attractive interaction potential proportional to r^α ($\alpha \neq 0$) or $\log r$ with the following conditions: (I) the moment of inertia is a non-zero constant, (II) the angular momentum is zero and (III) one body is on the centre of mass at an instant. We proved theorem 1: *motion which satisfies conditions (I)–(III) with equal masses under the potential $\alpha \neq -2, 2, 4$ is impossible*.

We solved explicitly motions which satisfy conditions (I)–(III) with equal masses under the potential $\alpha = 2$ or 4, and show that these motions do not have figure-eight shape and have collision. Since three-body figure-eight choreography satisfies conditions (II)–(III) with equal masses and has no collision, the Saari–Chenciner problem is solved.

We also proved theorem 2: *motion which satisfies conditions (I)–(III) with general masses under the Newtonian potential $\alpha = -1$ is impossible*.

This paper is organized as follows. In section 2, we clarify the consequences of conditions (I)–(III) for general masses and general α. Prescription of our proof of theorems 1 and 2 is given in this section. In section 3, we treat the case of equal masses. In section 3.1, a proof of theorem 1 is given. In section 3.2, we give motions explicitly which satisfy conditions (I)–(III) with equal masses under the potential $\alpha = 2$ or 4, and show that these solutions do not have figure-eight shape, and have collision. In section 4, we treat the case with general masses under the Newtonian potential $\alpha = -1$, and give a proof of theorem 2. Summary and discussions are given in section 5. Some algebraic details for section 3.1 are shown in the appendix.

2. Consequences of conditions (I)–(III)

In this section, we clarify the consequences of conditions (I)–(III) with general masses and general α, and give prescription of our proof of theorems 1 and 2.

Let us consider the three-body problem in three-dimensional space. Let m_i be masses of bodies $i = 1, 2, 3$, and let $\mathbf{r}_i(t)$ and $\mathbf{v}_i(t)$ be position and velocity vectors of them at time t, respectively. The moment of inertia with respect to the origin I, the kinetic energy K and the angular momentum L are defined as follows,

$$I = \frac{1}{2} \sum_i m_i \mathbf{r}_i^2 \tag{1}$$

$$K = \frac{1}{2} \sum_i m_i \mathbf{v}_i^2 \tag{2}$$

$$L = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i. \tag{3}$$

To treat the power law and logarithmic potentials uniformly, we use the following expression for the potential energy,

$$V_\alpha = \alpha^{-1} \sum_{i>j} m_i m_j r_{ij}^\alpha \qquad \text{for} \quad \alpha \neq 0$$

$$= \sum_{i>j} m_i m_j \log r_{ij} \qquad \text{for} \quad \alpha = 0 \tag{4}$$

where r_{ij} represents the mutual distance of body i and j, i.e., $r_{ij} = \sqrt{(\mathbf{r}_i - \mathbf{r}_j)^2}$. Note that the force $\mathbf{f}_i$ acting on the body i given by

$$\mathbf{f}_i = -\frac{\partial V_\alpha}{\partial \mathbf{r}_i} = m_i \sum_{j \neq i} m_j (\mathbf{r}_j - \mathbf{r}_i) r_{ji}^{\alpha-2} \tag{5}$$

is a continuous function of α and is an attractive force for all α. Non-existence of motions with constant moment of inertia under repulsive forces is obvious. See the comment on repulsive force in the end of section 5.

Without loss of generality, we can take the centre of mass to be the origin,

$$\sum_i m_i \mathbf{r}_i(t) = \mathbf{0} \tag{6}$$

the origin of time, $t = 0$, to be the instant of condition (III), and

$$\mathbf{r}_3(0) = \mathbf{0}. \tag{7}$$

Then the equations for the centre of mass (6), the first derivative of the moment of inertia (1) with respect to time and the zero angular momentum (3) at $t = 0$ yield

$$m_1 \mathbf{r}_1(0) + m_2 \mathbf{r}_2(0) = \mathbf{0} \tag{8}$$

$$m_1 \mathbf{r}_1(0) \cdot \mathbf{v}_1(0) + m_2 \mathbf{r}_2(0) \cdot \mathbf{v}_2(0) = 0 \tag{9}$$

$$m_1 \mathbf{r}_1(0) \times \mathbf{v}_1(0) + m_2 \mathbf{r}_2(0) \times \mathbf{v}_2(0) = \mathbf{0}. \tag{10}$$

Using equation (8), let $\mathbf{a} = m_1 \mathbf{r}_1(0) = -m_2 \mathbf{r}_2(0)$. Then the above equations become

$$\mathbf{a} \cdot (\mathbf{v}_1(0) - \mathbf{v}_2(0)) = 0 \tag{11}$$

$$\mathbf{a} \times (\mathbf{v}_1(0) - \mathbf{v}_2(0)) = \mathbf{0}. \tag{12}$$

Since $(\mathbf{a} \cdot \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b})^2 = (\mathbf{a}^2)(\mathbf{b}^2)$ holds for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$, equations (11) and (12) demand $\mathbf{a} = \mathbf{0}$ or $\mathbf{v}_1(0) = \mathbf{v}_2(0)$. If $\mathbf{a} = \mathbf{0}$ then $\mathbf{r}_i(0) = \mathbf{0}$ for all $i = 1, 2, 3$ and the moment of inertia at $t = 0$ is zero. This contradicts condition (I). Then, we can express variables at $t = 0$ as follows,

$$\mathbf{a} = m_1 \mathbf{r}_1(0) = -m_2 \mathbf{r}_2(0) \neq \mathbf{0} \qquad \mathbf{r}_3(0) = \mathbf{0} \tag{13}$$

$$\mathbf{v}_1(0) = \mathbf{v}_2(0) = -\mathbf{u} \qquad \mathbf{v}_3(0) = \frac{m_1 + m_2}{m_3} \mathbf{u}. \tag{14}$$

Therefore, motion under conditions (I)–(III) must be on a plane defined by $\mathbf{a}$ and $\mathbf{u}$. Here, it is well known that the three-body motion with zero angular momentum, condition (II), is always planar [9, 10]. Using the rotation and the scaling invariance of this system, we can take the

Cartesian component of these variables as follows,

$$\mathbf{r}_1(0) = \left(\frac{2m_2}{m_1 + m_2}, 0 \right) \tag{15}$$

$$\mathbf{r}_2(0) = \left(-\frac{2m_1}{m_1 + m_2}, 0 \right) \tag{16}$$

$$\mathbf{r}_3(0) = (0, 0) \tag{17}$$

$$\mathbf{u} = u(\cos\theta, \sin\theta) \qquad u > 0 \qquad 0 \leqslant \theta < 2\pi. \tag{18}$$

Then the kinetic and potential energies at $t = 0$ are given by

$$K(0) = \frac{(m_1 + m_2)(m_1 + m_2 + m_3)u^2}{2m_3} \tag{19}$$

and

$$\alpha V_\alpha(0) = m_1 m_2 2^\alpha + m_2 m_3 \left(\frac{2m_1}{m_1 + m_2} \right)^\alpha + m_3 m_1 \left(\frac{2m_2}{m_1 + m_2} \right)^\alpha \qquad \text{for} \quad \alpha \neq 0. \tag{20}$$

The second derivative of the moment of inertia with respect to time yields the Lagrange–Jacobi identity,

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K - \alpha V_\alpha = 2E - (2 + \alpha)V_\alpha \qquad \text{for} \quad \alpha \neq 0$$
$$= 2K - \sum_{i>j} m_i m_j = 2E - \sum_{i>j} m_i m_j - 2V_0 \qquad \text{for} \quad \alpha = 0 \tag{21}$$

where E represents the total energy $E = K + V_\alpha$. Thus the condition for the second derivative $\mathrm{d}^2 I/\mathrm{d}t^2 = 0$ yields

$$K = 2^{-1}\alpha V_\alpha \qquad \text{for} \quad \alpha \neq 0$$
$$= 2^{-1} \sum_{i>j} m_i m_j \qquad \text{for} \quad \alpha = 0. \tag{22}$$

Note that the right-hand side of the above equation is a continuous function of α. Equations (19), (20) and (22) determine the speed u in equation (18) for all α, as follows:

$$u^2 = m_3((m_1 + m_2)(m_1 + m_2 + m_3))^{-1}$$
$$\times \left(m_1 m_2 2^\alpha + m_2 m_3 \left(\frac{2m_1}{m_1 + m_2} \right)^\alpha + m_3 m_1 \left(\frac{2m_2}{m_1 + m_2} \right)^\alpha \right). \tag{23}$$

As shown above, conditions $I \neq 0$, $\mathrm{d}I/\mathrm{d}t = 0$, $\mathrm{d}^2 I/\mathrm{d}t^2 = 0$ and (II)–(III) at $t = 0$ determine the initial values with only one parameter θ in equation (18) left undetermined.

Higher order derivatives of $I = \text{const}$,

$$\frac{\mathrm{d}^{n+2} I}{\mathrm{d}t^{n+2}} = -(2 + \alpha)\frac{\mathrm{d}^n V_\alpha}{\mathrm{d}t^n} = 0 \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

do not produce any more conditions for $\alpha = -2$. On the other hand, for the case $\alpha \neq -2$ they give infinitely many conditions. We call these equations at $t = 0$

$$\frac{\mathrm{d}^n V_\alpha}{\mathrm{d}t^n}(0) = 0 \qquad \text{for} \quad n = 1, 2, 3, \ldots \tag{24}$$

the consistency conditions because these conditions must be satisfied by the initial values given above if motion with conditions (I)–(III) is consistent.

By virtue of the equation of motion, the differential operator d/dt acting on V_α is given by

$$\frac{d}{dt} = \sum_i \mathbf{v}_i \frac{\partial}{\partial \mathbf{r}_i} - \sum_i m_i^{-1} \frac{\partial V_\alpha}{\partial \mathbf{r}_i} \frac{\partial}{\partial \mathbf{v}_i}. \tag{25}$$

Using this expression and the initial values, we can calculate $d^n V_\alpha/dt^n$ at $t = 0$ up to any order we want. In the following sections we check the consistency conditions (24), and prove theorems 1 and 2.

3. The case with three equal masses

3.1. Inconstancy of moment of inertia with equal masses for $\alpha \neq -2, 2, 4$

In this section, we check the consistency conditions (24) with equal masses for $\alpha \neq -2$ and prove theorem 1.

We take $m_i = 1$ for all $i = 1, 2, 3$. Then the initial values in (15)–(20) are

$$\mathbf{r}_1(0) = (1, 0) \qquad \mathbf{r}_2(0) = (-1, 0) \qquad \mathbf{r}_3(0) = (0, 0) \tag{26}$$

$$\mathbf{v}_1(0) = \mathbf{v}_2(0) = -\mathbf{u} \qquad \mathbf{v}_3(0) = 2\mathbf{u} \qquad \mathbf{u} = u(\cos\theta, \sin\theta) \tag{27}$$

$$K(0) = 3u^2 \tag{28}$$

$$V_\alpha(0) = \frac{2^\alpha + 2}{\alpha} \qquad \text{for} \quad \alpha \neq 0. \tag{29}$$

From equation (23), the speed u is

$$u = \sqrt{(2^\alpha + 2)/6} \tag{30}$$

for all α, including $\alpha = 0$.

Let us check the consistency conditions (24). Since the time reversal of the initial values (26) and (27) is equivalent to the $180°$ rotation of this system around the origin and exchange of the index $1 \leftrightarrow 2$ and the potential V_α is invariant under this transformation, the potential V_α is invariant under the time reversal, i.e., $V_\alpha(\mathbf{r}_i(-t)) = V_\alpha(\mathbf{r}_i(t))$. Therefore, all odd-order derivatives at $t = 0$ vanish

$$\frac{d^n V_\alpha}{dt^n}(0) = 0 \qquad \text{for} \quad n = 1, 3, 5, \ldots. \tag{31}$$

The consistency condition for the second derivative gives

$$0 = \frac{d^2 V_\alpha}{dt^2}(0) = 2^{-1}(2 + 2^\alpha)(3(\alpha - 2)\cos(2\theta) - (2 + 2^\alpha - 3\alpha)). \tag{32}$$

If $\alpha = 2$, this equation is satisfied for all θ. For $\alpha \neq 2$, this equation yields

$$\cos(2\theta) = \frac{2 + 2^\alpha - 3\alpha}{3(\alpha - 2)} = \frac{2^\alpha - 2^2}{3(\alpha - 2)} - 1. \tag{33}$$

Note that the right-hand side is a monotonically increasing continuous function of α, is 1 at $\alpha = 4$ and is larger than -1 for all α. Thus, there is no solution of θ for $\alpha > 4$, i.e., there are no motions for $\alpha > 4$. For $\alpha \leqslant 4$ and $\alpha \neq 2$, the angle θ is given by the above equation. Especially, $\cos 2\theta = 1$ for $\alpha = 4$. Thus, initial values are completely determined for $\alpha \leqslant 4$ and $\alpha \neq 2$.

We can write down the consistency conditions for the fourth and sixth derivatives, applying the derivative operator d/dt given by equation (25) to V_α four or six times and substituting the

initial values given above as

$$\frac{d^4 V_\alpha}{dt^4}(0) = \frac{(2^\alpha + 2) f_4(\alpha, 2^\alpha)}{8(\alpha - 2)} = 0 \tag{34}$$

$$\frac{d^6 V_\alpha}{dt^6}(0) = \frac{(2^\alpha + 2) f_6(\alpha, 2^\alpha)}{32(\alpha - 2)^2} = 0 \tag{35}$$

where

$$f_4(x, y) = x^2(128 - 36y + 24y^2 + y^3) - 2xy(-112 + 62y + 5y^2)$$
$$+ 8(-32 - 38y + 13y^2 + 3y^3) \tag{36}$$

$$f_6(x, y) = x^4(6144 + 6496y - 1816y^2 + 60y^3 + 50y^4 + y^5)$$
$$- 4x^3(10496 + 6520y - 3676y^2 + 508y^3 + 266y^4 + 7y^5)$$
$$+ 4x^2(256 - 10288y - 15032y^2 + 1952y^3 + 1846y^4 + 71y^5)$$
$$- 16x(-5120 - 10840y - 9428y^2 - 148y^3 + 1186y^4 + 77y^5)$$
$$+ 64(-448 - 1596y - 1860y^2 - 299y^3 + 204y^4 + 30y^5). \tag{37}$$

One can easily verify that $\alpha = 2$ or 4 are the common roots of $f_4(\alpha, 2^\alpha) = 0$ and $f_6(\alpha, 2^\alpha) = 0$. Moreover $f_4(\alpha, 2^\alpha) = 0$ and $f_6(\alpha, 2^\alpha) = 0$ have another root $\alpha = -1.88\dots$ and $\alpha = -1.82\dots$, respectively. In the appendix, we prove rigorously the common root of $f_4(\alpha, 2^\alpha) = 0$ and $f_6(\alpha, 2^\alpha) = 0$ are only $\alpha = 2, 4$. Therefore, the existence of motion which satisfies conditions (I)–(III) with equal masses is not consistent for $\alpha \neq -2, 2, 4$.

3.2. Motions for $\alpha = 2, 4$

In this section, we give motions explicitly which satisfy conditions (I)–(III) with equal masses under the potential $\alpha = 2$ and 4. Then, we discuss the origin of these solutions from general framework.

For $\alpha = 2$, equation (30) gives $u = 1$. The initial values are

$$\mathbf{r}_1(0) = (1, 0) \qquad \mathbf{r}_2(0) = (-1, 0) \qquad \mathbf{r}_3(0) = (0, 0) \tag{38}$$

$$\mathbf{v}_1(0) = \mathbf{v}_2(0) = -\mathbf{u} \qquad \mathbf{v}_3(0) = 2\mathbf{u} \qquad \mathbf{u} = u(\cos\theta, \sin\theta) \tag{39}$$

with $u = 1$. The equation of motion is

$$\frac{d^2 \mathbf{r}_i}{dt^2}(t) = \sum_{j \neq i}(\mathbf{r}_j(t) - \mathbf{r}_i(t)) = -3\mathbf{r}_i(t). \tag{40}$$

Here, we have used $\sum_i \mathbf{r}_i(t) = 0$. The solution is given by

$$\mathbf{r}_1(t) = (\cos(\sqrt{3}t), 0) - \tfrac{1}{2}\mathbf{r}_3(t) \tag{41}$$

$$\mathbf{r}_2(t) = (-\cos(\sqrt{3}t), 0) - \tfrac{1}{2}\mathbf{r}_3(t) \tag{42}$$

$$\mathbf{r}_3(t) = \tfrac{2}{\sqrt{3}}\sin(\sqrt{3}t)(\cos\theta, \sin\theta) \tag{43}$$

with arbitrary angle θ. One can easily verify that the moment of inertia is constant $I = 2^{-1}\sum_i \mathbf{r}_i^2(t) = 1$. Obviously, the shape of this motion is not figure-eight. This is because figure-eight must have two different periods for major and minor axes, while this

potential is for an isotropic harmonic oscillator. Since $\mathbf{r}_1 - \mathbf{r}_2 = (2\cos(\sqrt{3}t), 0)$, the bodies 1 and 2 collide at $t = \pi/(2\sqrt{3})$ for any angle θ.

For $\alpha = 4$, the initial values are the same as equations (38) and (39) with $u = \sqrt{3}$ and

$$\cos(2\theta) = 1. \tag{44}$$

The motion is therefore one dimensional. We take the angle $\theta = 0$. Motion with $\theta = \pi$ is equivalent to the time reversal motion of $\theta = 0$. We write $\mathbf{r}_i(t) = (x_i(t), 0)$ and $\mathbf{v}_i(t) = (v_i(t), 0)$. Initial conditions are

$$x_1(0) = 1 \qquad x_2(0) = -1 \qquad x_3(0) = 0 \tag{45}$$

$$v_1(0) = v_2(0) = -\sqrt{3} \qquad v_3(0) = 2\sqrt{3}. \tag{46}$$

The equation of motion is

$$\frac{d^2 x_i}{dt^2}(t) = \sum_{j \neq i} (x_j(t) - x_i(t))^3 = (x_j + x_k - 2x_i) \left(\sum_\ell x_\ell^2 - \sum_{\ell > m} x_\ell x_m \right) \tag{47}$$

with $(i, j, k) = (1, 2, 3), (2, 3, 1)$ or $(3, 1, 2)$. Since $\sum_i x_i(t) = 0$, this equation is reduced into [11]

$$\frac{d^2 x_i}{dt^2}(t) = -\frac{9}{2} x_i \left(\sum_j x_j^2 \right). \tag{48}$$

Therefore, if the moment of inertia is constant $I = 2^{-1} \sum_i x_i^2(t) = 1$, the equation of motion is equivalent to that of a harmonic oscillator

$$\frac{d^2 x_i}{dt^2}(t) = -9x_i. \tag{49}$$

The solution is given by

$$x_1(t) = \frac{2}{\sqrt{3}} \sin\left(3t + \frac{2\pi}{3}\right) \tag{50}$$

$$x_2(t) = \frac{2}{\sqrt{3}} \sin\left(3t - \frac{2\pi}{3}\right) \tag{51}$$

$$x_3(t) = \frac{2}{\sqrt{3}} \sin(3t). \tag{52}$$

One can easily verify that the moment of inertia of this solution is constant $I = 1$, and that the bodies 1 and 3 collide at $t = \pi/18$. Since this motion is one dimensional, this motion is not figure-eight too.

The origin of the above solutions are as follows [11, 12]. For $\alpha = 2$, let us consider three-dimensional motions with general masses m_i. The fact that the centre of mass is at the origin implies

$$I = \frac{1}{2} \sum_i m_i \mathbf{r}_i^2 = \frac{1}{2M} \sum_i m_i m_j (\mathbf{r}_i - \mathbf{r}_j)^2 = \frac{V_2}{M}.$$

Here we write $M = \sum_i m_i$. Then the Lagrange–Jacobi identity (21) yields

$$\frac{d^2 I}{dt^2} = 2E - 4MI.$$

For $\alpha = 4$, let us consider one-dimensional motions with equal masses. The situation is similar due to the identity

$$\frac{1}{2}\left(\sum_{i>j}(x_i - x_j)^2\right)^2 = \sum_{i>j}(x_i - x_j)^4.$$

Then the Lagrange–Jacobi identity yields

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2E - 27I^2.$$

Therefore $\mathrm{d}I/\mathrm{d}t = 0$ and $\mathrm{d}^2 I/\mathrm{d}t^2 = 0$ is sufficient to prove $I = \text{constant}$ for both cases. This is the reason why higher derivatives of I do not give any conditions for initial values as shown in section 3.1.

4. The case with general masses for $\alpha = -1$

In this section, we check the consistency conditions (24) with general masses under the Newtonian gravity $\alpha = -1$, and prove theorem 2.

From equation (23), the speed u is given by

$$u = \sqrt{\frac{m_3\left(m_1^2 m_2^2 + \left(m_1^3 + m_1^2 m_2 + m_1 m_2^2 + m_2^3\right)m_3\right)}{2m_1 m_2(m_1 + m_2)(m_1 + m_2 + m_3)}}. \tag{53}$$

And consistency condition for the first derivative gives

$$0 = \frac{\mathrm{d}V_{-1}}{\mathrm{d}t}(0) = -\frac{\left(m_1^3 - m_2^3\right)(m_1 + m_2)^2(m_1 + m_2 + m_3)u\cos\theta}{4m_1^2 m_2^2}. \tag{54}$$

This is satisfied if $m_1 = m_2$ or $\cos\theta = 0$.

(1) For the case $m_1 = m_2$: Let $m_1 = m_2 = m$ and $m_3 = \mu m$. Then the speed u becomes

$$u = \frac{1}{2}\sqrt{\frac{m\mu(1 + 4\mu)}{2 + \mu}}. \tag{55}$$

Since $m_1 = m_2$, the time reversal is equivalent to $180°$ rotation around the origin and exchange the index $1 \leftrightarrow 2$. Therefore $V_{-1}(\mathbf{r}_i(-t)) = V_{-1}(\mathbf{r}_i(t))$ and the consistency conditions for odd-order derivatives are satisfied $\mathrm{d}^n V_{-1}/\mathrm{d}t^n(0) = 0$ for $n = 1, 3, 5, \ldots$. The condition for the second derivative gives

$$0 = \frac{\mathrm{d}^2 V_{-1}}{\mathrm{d}t^2}(0) = -8^{-1}m^3(1 + 4\mu)\left(5 + 6\mu + 6(2 + \mu)\cos(2\theta)\right). \tag{56}$$

This condition is satisfied by the angle

$$\cos 2\theta = -\frac{5 + 6\mu}{12 + 6\mu}.$$

And the condition for the fourth derivative,

$$0 = \frac{\mathrm{d}^4 V_{-1}}{\mathrm{d}t^4}(0) = -\frac{m^4(1 + 4\mu)(-1597 - 1576\mu + 432\mu^2)}{384\mu}, \tag{57}$$

can be satisfied if

$$\mu = \frac{197 + 14\sqrt{418}}{108}.$$

But the sixth derivative is always negative,

$$\frac{d^6 V_{-1}}{dt^6}(0) = -\frac{m^5}{6144\mu^2}(315165 + 2686088\mu + 6911872\mu^2$$

$$+ 4944512\mu^3 + 443136\mu^4 + 110592\mu^5). \tag{58}$$

Therefore, $V_{-1}(t) =$ const is not consistent in this case.

(2) For the case $m_1 \neq m_2$ and $\cos\theta = 0$: In this case, the time reversal is equivalent to reflection of the y-axis, $y \leftrightarrow -y$. Therefore $V_{-1}(\mathbf{r}_i(-t)) = V_{-1}(\mathbf{r}_i(t))$ and the consistency conditions for odd-order derivatives are satisfied $d^n V_{-1}/dt^n(0) = 0$ for $n = 1, 3, 5, \ldots$. The condition for the second derivative gives quadratic equation for m_3

$$0 = \frac{d^2 V_{-1}}{dt^2}(0) = -\frac{(m_1 + m_2)(c_2 m_3^2 - c_1 m_3 - c_0)}{16 m_1^3 m_2^3} \tag{59}$$

with

$$c_2 = (m_1 - m_2)^2 (m_1 + m_2)^2 (m_1^2 + m_1 m_2 + m_2^2) \tag{60}$$

$$c_1 = 2m_1 m_2 (m_1 + m_2)(m_1^4 + m_1^3 m_2 + 3m_1^2 m_2^2 + m_1 m_2^3 + m_2^4) \tag{61}$$

$$c_0 = m_1 m_2 (m_1^6 + 2m_1^5 m_2 + m_1^4 m_2^2 - m_1^3 m_2^3 + m_1^2 m_2^4 + 2m_1 m_2^5 + m_2^6) > 0. \tag{62}$$

This condition is satisfied if

$$m_3 = \frac{c_1 + \sqrt{c_1^2 + 4c_0 c_2}}{2c_2}. \tag{63}$$

But the fourth derivative has the following form:

$$\frac{d^4 V_{-1}}{dt^4}(0) = -\frac{3(m_1 + m_2)^2 (p_0(m_1, m_2) + p_1(m_1, m_2)\sqrt{m_1 m_2 \Omega})}{q(m_1, m_2)} \tag{64}$$

where

$$p_0 = 7m_1^{22} + 74m_1^{21}m_2 + 321m_1^{20}m_2^2 + 955m_1^{19}m_2^3 + 2335m_1^{18}m_2^4$$
$$+ 4925m_1^{17}m_2^5 + 9261m_1^{16}m_2^6 + 15383m_1^{15}m_2^7 + 22843m_1^{14}m_2^8$$
$$+ 29992m_1^{13}m_2^9 + 35297m_1^{12}m_2^{10} + 37102m_1^{11}m_2^{11}$$
$$+ 35297m_1^{10}m_2^{12} + 29992m_1^9 m_2^{13} + 22843m_1^8 m_2^{14} + 15383m_1^7 m_2^{15}$$
$$+ 9261m_1^6 m_2^{16} + 4925m_1^5 m_2^{17} + 2335m_1^4 m_2^{18} + 955m_1^3 m_2^{19}$$
$$+ 321m_1^2 m_2^{20} + 74m_1 m_2^{21} + 7m_2^{22} \tag{65}$$

$$p_1 = 21m_1^{16} + 136m_1^{15}m_2 + 457m_1^{14}m_2^2 + 1104m_1^{13}m_2^3 + 2049m_1^{12}m_2^4$$
$$+ 3284m_1^{11}m_2^5 + 4510m_1^{10}m_2^6 + 5516m_1^9 m_2^7 + 5830m_1^8 m_2^8$$
$$+ 5516m_1^7 m_2^9 + 4510m_1^6 m_2^{10} + 3284m_1^5 m_2^{11} + 2049m_1^4 m_2^{12}$$
$$+ 1104m_1^3 m_2^{13} + 457m_1^2 m_2^{14} + 136m_1 m_2^{15} + 21m_2^{16} \tag{66}$$

$$\Omega = m_1^{10} + 2m_1^9 m_2 + m_1^8 m_2^2 + 4m_1^7 m_2^3 + 9m_1^6 m_2^4 + 15m_1^5 m_2^5$$
$$+ 9m_1^4 m_2^6 + 4m_1^3 m_2^7 + m_1^2 m_2^8 + 2m_1 m_2^9 + m_2^{10} \tag{67}$$

and

$$q = 128(m_1 - m_2)^4 (m_1^2 + m_1 m_2 + m_2^2)^3 (m_1 m_2)^2$$
$$\times ((m_1^4 + m_1^3 m_2 + 3m_1^2 m_2^2 + m_1 m_2^3 + m_2^4)m_1 m_2 + \sqrt{m_1 m_2 \Omega}). \tag{68}$$

Thus $d^4 V_{-1}/dt^4(0) < 0$. Therefore, $V_{-1}(t) =$ const is not consistent in this case too.

5. Summary and discussions

In this paper, we have solved Saari–Chenciner's problem: *the moment of inertia of the three-body figure-eight choreography under the attractive potential V_α defined in (4) stays constant if and only if $\alpha = -2$*. Saari–Chenciner's problem for $\alpha = -1$ is a tiny piece of Saari's conjecture, because a *relative equilibrium* of the three bodies yields the non-zero angular momentum and contradicts the zero angular momentum of the figure-eight choreography. On the other hand, since Saari–Chenciner's problem states the motion for all α, it is considered to be a partial extension of Saari's conjecture.

Though the three-body figure-eight choreography with $\alpha = -2$ having constant moment of inertia [3, 5] is not given analytically, it will be obtained numerically from the set of initial conditions at $t = 0$ given in equations (26), (27) and (30), i.e.,

$$\mathbf{r}_1(0) = (1, 0) \qquad \mathbf{r}_2(0) = (-1, 0) \qquad \mathbf{r}_3(0) = (0, 0)$$

and

$$\mathbf{v}_1(0) = \mathbf{v}_2(0) = -\tfrac{1}{2}\sqrt{\tfrac{3}{2}}(\cos\theta, \sin\theta) \qquad \mathbf{v}_3(0) = \sqrt{\tfrac{3}{2}}(\cos\theta, \sin\theta).$$

Here suitable values of θ should be chosen. The analytical method how to determine the values of θ is not known. If a θ_0 gives a figure-eight then $-\theta_0$ and $\pi \pm \theta_0$ give the same figure-eight. So we have four θ for one figure-eight. Uniqueness of figure-eight is still unproved.

In order to solve Saari–Chenciner's problem, we have considered the motion which satisfies conditions (II) and (III) instead of the three-body figure-eight choreography. The set of initial conditions of the motion, $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, at the instant of condition (III), $t = 0$, has been written by only one parameter θ in equation (18). The set has been determined by conditions (II), (III), $I \neq 0$, $dI/dt = 0$ and $d^2I/dt^2 = 0$ at $t = 0$.

Since we have considered the motion under conditions (II) and (III), we have obtained theorem 1 as a by-product. It is applicable to wide class of motion more than the figure-eight choreography. For example, though the H3 orbit found by Simó [6] is not the figure-eight choreography because each particle runs in different figure-eight orbit, it satisfies conditions (II), (III) and $I \neq 0$. Therefore, the H3 orbit cannot have constant moment of inertia by theorem 1.

Theorem 1 states that there may exist motions for $\alpha = 2, 4$ other than $\alpha = -2$. In connection with Saari–Chenciner's problem, we have given the explicit solutions for $\alpha = 2, 4$ and have shown that they are never the figure-eight choreography because they all have collisions.

If we consider a wider class of interaction potentials other than the power law including log potential, the figure-eight choreography with constant moment of inertia is possible. For example, under the artificial potential $1/2 \log r - \sqrt{8}/3r^2$, we find a three-body choreography on the lemniscate, a kind of analytical figure-eight. This motion has a constant moment of inertia [13].

As we noted in section 2, we comment on the non-existence of the motion under the repulsive potential $-V_\alpha$ for the general mass three-body system for all α. This is almost obvious but can be proved since equation (21) becomes

$$\frac{d^2 I}{dt^2} = 2K + \alpha V_\alpha > 0$$

for $\alpha \neq 0$ and

$$\frac{d^2 I}{dt^2} = 2K + \sum_{i > j} m_i m_j > 0$$

for $\alpha = 0$ by replacing V_α to $-V_\alpha$.

For the three-body system with general masses, the same analysis will be possible but it will become more complex. We then have done the analysis only for the realistic potential, the Newtonian potential $\alpha = -1$ and have obtained theorem 2, which states that the motion having constant moment of inertia is impossible. This is also consistent with Saari's conjecture. Though theorem 2 is still a tiny piece of Saari's Conjecture, it will be extended to arbitrary α. Actually for the equal mass system we could obtain theorem 1 and could apply it to solve Saari–Chenciner's problem. Extension of theorem 2 for all $\alpha \neq -2$ is left for future work.

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Appendix. Common roots of $d^4 V_\alpha / dt^4(0) = 0$ and $d^6 V_\alpha / dt^6(0) = 0$

In this appendix, we prove that the common roots of $f_4(\alpha, 2^\alpha) = 0$ and $f_6(\alpha, 2^\alpha) = 0$ are only $\alpha = 2, 4$. Functions $f_4(x, y)$ and $f_6(x, y)$ are defined by equations (36) and (37).

By the Euclidean algorithm for polynomials of two variables [14], we can find polynomials $L(x, y)$, $M(x, y)$ and resultant $R(y)$, which satisfy

$$L(x, y) f_6(x, y) - M(x, y) f_4(x, y) = R(y). \tag{A.1}$$

Actually,

$$
\begin{aligned}
L(x, y) = {} & x(268435456 + 5704253440y - 4900519936y^2 - 10788732928y^3 \\
& + 1665391872y^4 - 2044335168y^5 - 339308448y^6 - 16071552y^7 \\
& + 3559728y^8 - 160420y^9 + 31166y^{10} + 2482y^{11} + 31y^{12}) \\
& - 4(67108864 - 2313158656y - 803405824y^2 + 6805321728y^3 \\
& + 4795789440y^4 - 1930678368y^5 - 379246848y^6 - 11684784y^7 \\
& + 1953024y^8 + 113414y^9 - 4550y^{10} + 2707y^{11} + 45y^{12}) \tag{A.2}
\end{aligned}
$$

$$
\begin{aligned}
M(x, y) = {} & x^3(12884901888 + 291051143168y + 129899692032y^2 \\
& - 865502298112y^3 - 665337083904y^4 + 113529684992y^5 \\
& + 12851181696y^6 - 4933789824y^7 - 907026720y^8 + 25947264y^9 \\
& + 2714152y^{10} - 299016y^{11} + 79330y^{12} + 3288y^{13} + 31y^{14}) \\
& - 2x^2(50465865728 + 773060558848y - 95409930240y^2 \\
& - 2156632178688y^3 - 752601498624y^4 + 570103937280y^5 \\
& - 53961508992y^6 - 60918928512y^7 - 6690174624y^8 + 59713360y^9 \\
& + 43040792y^{10} - 1087920y^{11} + 657030y^{12} + 33936y^{13} + 369y^{14}) \\
& + 8x(14495514624 - 274861129728y - 257627258880y^2 + 562210586624y^3 \\
& + 1106047222784y^4 + 651747223040y^5 - 45793043520y^6 - 80250740736y^7 \\
& - 9184728480y^8 - 48378720y^9 + 67514796y^{10} \\
& - 1554720y^{11} + 605392y^{12} + 54856y^{13} + 715y^{14}) \\
& - 32(1006632960 - 19713228800y - 94432198656y^2 - 4409028608y^3 \\
& + 275213442304y^4 + 281906870976y^5 + 33707210784y^6 \\
& - 29192050368y^7 - 4054720752y^8 - 83636004y^9 + 22319090y^{10} \\
& + 845634y^{11} + 17501y^{12} + 28166y^{13} + 450y^{14}) \tag{A.3}
\end{aligned}
$$

and

$$R(y) = -512(-16 + y)(-4 + y)^4(2 + y)^2 f(y) \tag{A.4}$$

with

$$f(y) = -65536 - 10276864y - 5027392y^2 + 25146656y^3 + 27552272y^4$$
$$+ 7538528y^5 - 180256y^6 - 27646y^7 + 944y^8 + 21y^9. \tag{A.5}$$

From equation (A.1), it is obvious that common roots of $f_4(\alpha, 2^\alpha) = 0$ and $f_6(\alpha, 2^\alpha) = 0$ are roots of $R(y) = 0$ with $y = 2^\alpha$.

The obvious roots of $R(y) = 0$ are $y = 2^\alpha = 4$ and 16. The Sturm's theorem [14, 15] shows that $f(y) = 0$ has only one root for $y = 2^\alpha > 0$. Since

$$f\left(\frac{1}{2}\right) = -\frac{697813379}{512} < 0$$

and

$$f\left(\frac{1}{\sqrt{2}}\right) = 4286363 + \frac{66842265}{16\sqrt{2}} > 0$$

the positive root y_0 is in the interval $1/2 < y_0 < 1/\sqrt{2}$. Therefore roots of $R(2^\alpha) = 0$ are $\alpha = 2, 4$ and α_0 with $-1 < \alpha_0 < -1/2$.

But we can show that $f_4(\alpha, 2^\alpha) = 0$ does not have root between $-1 < \alpha_0 < -1/2$ as follows. Let us introduce the new variable $\beta = -\alpha$, and two monotonically increasing functions $g^{(+)}(\beta)$ and $g^{(-)}(\beta)$,

$$f_4(\alpha, 2^\alpha) = 2^{3\alpha} g^{(+)}(\beta) - g^{(-)}(\beta) \tag{A.6}$$

$$g^{(+)}(\beta) = \beta^2(128 \times 2^{3\beta} + 24 \times 2^\beta + 1) + 2\beta(62 \times 2^\beta + 5) + 8(13 \times 2^\beta + 3) \tag{A.7}$$

$$g^{(-)}(\beta) = 36\beta^2 \times 2^{2\beta} + 224\beta \times 2^{2\beta} + 16(16 \times 2^{3\beta} + 19 \times 2^{2\beta}). \tag{A.8}$$

If α_0 is a root of $f_4(\alpha, 2^\alpha) = 0$, $\beta_0 = -\alpha_0$ satisfies $g^{(+)}(\beta_0) = g^{(-)}(\beta_0)$ with $\frac{1}{2} < \beta_0 < 1$. Since functions $g^{(\pm)}(\beta)$ are monotonically increasing functions, we get

$$g^{(-)}\left(\tfrac{1}{2}\right) < g^{(-)}(\beta_0) = g^{(+)}(\beta_0) < g^{(+)}(1). \tag{A.9}$$

But actual values are

$$g^{(-)}\left(\tfrac{1}{2}\right) = 850 + 512\sqrt{2} > 1566$$

and

$$g^{(+)}(1) = 1563.$$

This is a contradiction.

Thus, we proved that common roots of $f_4(\alpha, 2^\alpha) = 0$ and $f_6(\alpha, 2^\alpha) = 0$ are only $\alpha = 2, 4$.

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